

# Geometric properties of QCD string from Willmore functional

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Received 5 September 2000; received in revised form 20 October 2000

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## Abstract

The extremum of the Willmore-like functional for  $m$ -dimensional Riemannian manifold immersed in  $d$ -dimensional Riemannian manifold under normal variations is studied and various cases of interest are examined. This study is used to relate the parameters of QCD string action, including the Polyakov–Kleinert extrinsic curvature action, with the geometric properties of the world sheet. The world sheet has been shown to have *negative stiffness* on the basis of the geometric considerations. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 51P05; 53C42; 53C80; 81V05

Subj. Class.: Strings

Keywords: Willmore functional; Immersion in Riemann space; QCD string action; Extrinsic curvature

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## 1. Introduction

QCD strings is a string theory in four dimensions. It has been realized by Polyakov [1,19] and independently by Kleinert [2,20,21] that for QCD strings, added extrinsic curvature action to the usual Nambu–Goto (NG) area term is appropriate. In particular the theory with extrinsic curvature action alone has been shown to be asymptotically free [1–3,19–21] — a feature relevant to describe QCD. By considering the 1-loop multi-instanton effects in the theory of two-dimensional world sheet in  $R^3$  and  $R^4$ , the grand partition function has been found to be that of a two-dimensional modified Coulomb gas system with long range order in the infra-red region and, in plasma phase in the ultra-violet region [4]. The above result uses the running coupling constant and the string world sheet is stable against small fluctuations

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along the normal (transverse) directions in the infra-red region and avoids crumpling. Thus the Polyakov–Kleinert string provides a relevant description of colour flux tubes between quarks in QCD.

In order to remove the unphysical ghost poles and to realize a lowest energy state, Kleinert and Chervyakov [5,22] recently proposed a *new* string model with *negative* sign for the extrinsic curvature action, i.e., they *hypothesize* negative *stiffness* for the gluonic flux tubes, inspired by properties of magnetic flux tubes in Type-II superconductor and of Nielsen–Olesen vortices in relativistic gauge models. They propose an effective string action as

$$S = \frac{c-1}{2} M^2 \int d^2\xi \sqrt{g} g^{\alpha\beta} \nabla_\alpha X^\mu(\xi) \frac{1}{c - e^{\nabla^2/\mu^2}} \nabla_\beta X^\mu(\xi),$$

where  $X^\mu(\xi)$ ;  $\mu = 1, 2, 3, 4$  are the world sheet coordinates,  $\xi_1, \xi_2$  are the local isothermal coordinates on the surface,  $g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\mu$  is the induced metric (first fundamental form) on the world sheet,  $\nabla_\alpha$  the covariant derivative on the surface,  $M$  the dimensionfull (mass dimension) constant and  $c$  is a dimensionless constant. The propagator from the quadratic part in  $X^\mu$  in momentum space [5,22] is

$$G(k^2) = \frac{1}{c-1} \frac{c - e^{-k^2/\mu^2}}{k^2},$$

and for small momentum, this is

$$G(k^2) \simeq \frac{1 + k^2/\Lambda^2}{k^2}$$

with  $\Lambda^2 = (c-1)/\mu^2$ . This has a single pole at  $k^2 = 0$  with negative stiffness  $\alpha_0 = -\Lambda^2/M^2$ , in contrast to the propagator in Polyakov–Kleinert model  $1/(k^2(1+k^2)/\Lambda^2)$ , which has unphysical pole at  $k^2 = -\Lambda^2$  and which has positive stiffness of  $\Lambda^2/M^2$ . Approximating the full propagator by its low momentum expression, Kleinert and Chervyakov [5,22] have proposed an action at low momentum region as

$$S_{KC} = \frac{1}{2} M^2 \int d^2\xi \sqrt{g} g^{\alpha\beta} \nabla_\alpha X^\mu \frac{1}{1 - (\nabla^2/\Lambda^2)} \nabla_\beta X^\mu. \tag{1}$$

Such an action (1) has the the high temperature behaviour as that of large-N QCD [6]. The *negative* extrinsic curvature term can be seen from (1) by expanding the non-local term using the Gauss equation

$$\nabla_\alpha \nabla_\beta X^\mu = H_{\alpha\beta}^i N^{i\mu}, \tag{2}$$

where  $i = 1, 2$  and  $H_{\alpha\beta}^i$  are the components of the extrinsic curvature (second fundamental form) along the two normals  $N^{i\mu}$  to the world sheet, and the Weingarten equation [4]

$$\nabla_\alpha N^{i\mu} = -H_\alpha^{i\gamma} \partial_\gamma X^\mu, \tag{3}$$

where the covariant derivative  $\nabla_\alpha$  in (3) incorporates the connection in the normal frame as well. By expanding  $(1 - (\nabla^2/\Lambda^2))^{-1}$  in (1) and realizing  $X^\mu$  is a scalar on the world sheet,

(1) can be written as

$$S_{\text{KC}} = \frac{1}{2} M^2 \int d^2\xi \sqrt{g} \left\{ 2 + \frac{1}{\Lambda^2} g^{\alpha\beta} \partial_\alpha X^\mu g^{\gamma\delta} \nabla_\gamma (\nabla_\delta \partial_\beta X^\mu) - \dots \right\},$$

where we have retained up to the  $1/\Lambda^2$  term for illustration. Upon using (2) and (3) and the fact  $\partial_\alpha X^\mu N^{i\mu} = 0$ , the above expression simplifies to

$$S_{\text{KC}} = \frac{1}{2} M^2 \int d^2\xi \sqrt{g} \left\{ 2 - \frac{1}{\Lambda^2} H^{i\alpha\beta} H^i_{\alpha\beta} \right\}.$$

But then,

$$H^{i\alpha\beta} H^i_{\alpha\beta} = 4|H|^2 + R,$$

where  $|H|^2 = H^i H^i$  with  $H^i = \frac{1}{2} g^{\alpha\beta} H^i_{\alpha\beta}$  and  $R$  the scalar curvature of the world sheet. In view of this, the expression for  $S_{\text{KC}}$  becomes

$$S_{\text{KC}} \simeq M^2 \int \sqrt{g} d^2\xi - \frac{2M^2}{\Lambda^2} \int \sqrt{g} |H|^2 d^2\xi - \frac{M^2}{\Lambda^2} \int \sqrt{g} R d^2\xi + \dots \tag{4}$$

In the above  $M^2$  plays the role of string tension. The extrinsic curvature action (the second term in (4)) has *negative* stiffness. The third term is just the Euler characteristic of the surface which is a topological invariant action. It is clear from (1) that unphysical poles can be avoided by appealing to surfaces with negative stiffness.

It will be worthwhile to examine whether the negative stiffness is favoured from purely geometric considerations of the surface. In this context, the Willmore surfaces which extremize the Willmore functional [7]

$$S_{\text{W}} = \int \sqrt{g} |H|^2 d^2\xi,$$

become relevant. It is the purpose of this paper to first consider general Willmore functional for  $m$ -dimensional manifold immersed in  $d$ -dimensional Riemannian space ( $m < d$ ) and study various cases of interest. Then using the results, we compare the classical equation of motion for (4) with immersion in flat space, with that of the Willmore functional for immersion in a Riemannian space, thereby showing the effects of the NG term in (4) could be accounted for by considerations of the Willmore functional in a curved space.

The extremum of Willmore functional for hypersurfaces in Euclidean space ( $E^3$ ) has been dealt with in detail by Willmore [7] and Chen [8] for  $m$ -dimensional oriented closed hypersurface in Euclidean space  $E^{m+1}$ . Willmore and Jhaveri [9] extended to  $m$ -dimensional manifold immersed as a hypersurface of a general  $(m+1)$ -dimensional Riemannian manifold and Weiner [10] to that of two-dimensional surface in a general Riemannian manifold. The geometric problem of the isometric immersion of a Riemannian manifold,  $(M, g)$ , in another Riemannian manifold,  $(\tilde{M}, \tilde{g})$  was first considered by Pedit and Willmore [11], by generalizing the definition of the Willmore functional and evaluating the Euler equation for the extremum of this functional. We here revisit this problem, explicitly derive the Euler equation under normal variations and then consider various cases of interest. As an

application of this study, we will compare the equation of motion of Willmore functional for two-dimensional surface immersed in  $d$ -dimensional space with the classical equation of motion of QCD string to relate the QCD string parameters, namely, the string tension and stiffness parameter to the geometrical properties of the surface.

## 2. Effects due to normal variations

For an  $m$ -dimensional manifold  $\Sigma$  immersed in a  $d$ -dimensional ( $d > m$ ) Riemannian manifold  $\Sigma'$  with metric  $h_{\mu\nu}$  ( $\mu, \nu = 1, 2, \dots, d$ ), we have the induced metric on  $\Sigma$  as

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu h_{\mu\nu}, \tag{5}$$

where the indices  $\alpha, \beta$  take values  $1, 2, \dots, m$  and  $X^\mu = X^\mu(\xi_1, \xi_2, \dots, \xi_m)$  with  $\xi_\alpha$ 's as coordinates on  $\Sigma$ . There are  $(d - m)$  unit normals at a point  $P \in \Sigma$ , denoted by  $N^{i\mu}$  ( $i = 1, 2, \dots, (d - m)$ ), chosen to satisfy

$$\begin{aligned} N^{i\mu} N^{j\nu} h_{\mu\nu} &= \delta^{ij}, & \partial_\alpha X^\mu N^{i\nu} h_{\mu\nu} &= 0 \\ \forall i &= 1, 2, \dots, (d - m), & \forall \alpha &= 1, 2, \dots, m. \end{aligned} \tag{6}$$

Repeated indices will be appropriately summed over in this paper. The equation of Gauss [12] for  $\Sigma$ ,

$$\nabla_\alpha \nabla_\beta X^\mu \equiv \partial_\alpha \partial_\beta X^\mu - \Gamma_{\alpha\beta}^\gamma \partial_\gamma X^\mu + \tilde{\Gamma}_{\nu\rho}^\mu \partial_\alpha X^\nu \partial_\beta X^\rho = H_{\alpha\beta}^i N^{i\mu}, \tag{7}$$

defines the *second fundamental form*  $H_{\alpha\beta}^i$ .  $\Gamma_{\alpha\beta}^\gamma$  and  $\tilde{\Gamma}_{\nu\rho}^\mu$  are the connections on  $\Sigma$  and  $\Sigma'$  determined by  $g_{\alpha\beta}$  and  $h_{\mu\nu}$ , respectively. The  $(d - m)$  normals  $N^{i\mu}$  satisfy the Weingarten equation

$$\nabla_\alpha N^{i\mu} \equiv \partial_\alpha N^{i\mu} + \tilde{\Gamma}_{\sigma\rho}^\mu \partial_\alpha X^\sigma N^{i\rho} - A_\alpha^{ij} N^{j\mu} = -H_{\alpha\beta}^{i\beta} \partial_\beta X^\mu, \tag{8}$$

where  $A_\alpha^{ij} = N^{j\nu} (\partial_\alpha N^{i\mu} + \tilde{\Gamma}_{\sigma\rho}^\mu \partial_\alpha X^\sigma N^{i\rho}) h_{\mu\nu}$  is the  $m$ -dimensional gauge field or connection in the normal bundle. We need the Gauss equation [12]

$$\tilde{R}_{\mu\nu\rho\sigma} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \partial_\delta X^\sigma = R_{\alpha\beta\gamma\delta} + H_{\beta\gamma}^i H_{\alpha\delta}^i - H_{\beta\delta}^i H_{\alpha\gamma}^i, \tag{9}$$

where  $R_{\alpha\beta\gamma\delta}$  and  $\tilde{R}_{\mu\nu\rho\sigma}$  are the Riemann symbols of the first kind for  $\Sigma$  and  $\Sigma'$ , respectively. We introduce the mean curvature  $H^i$  (there are  $(d - m)$  such quantities) by

$$H^i = \frac{1}{m} g^{\alpha\beta} H_{\alpha\beta}^i. \tag{10}$$

The variations of the surface can be described by the variations of  $X^\mu(\xi_1, \dots, \xi_m)$  as  $X^\mu(\xi_1, \dots, \xi_m) + \delta X^\mu(\xi_1, \dots, \xi_m)$ . In general  $\delta X^\mu = \phi^i N^{i\mu} + \partial_\alpha X^\mu \eta^\alpha$ , comprising of variations along  $m$  tangent directions and  $(d - m)$  normals. The tangential variations are related to the structure equations [13]. So, we consider only the normal variations and accordingly,

$$\delta X^\mu = \phi^i N^{i\mu}. \tag{11}$$

Using (5) it follows for normal variations

$$\delta\sqrt{g} = -m\sqrt{g}\phi^i H^i, \quad \delta g^{\alpha\beta} = 2\phi^i H^{i\alpha\beta}. \tag{12}$$

From (10) it follows

$$\delta H^i = \frac{2}{m}\phi^k H^{k\alpha\beta} H_{\alpha\beta}^i + \frac{1}{m}g^{\alpha\beta}\delta H_{\alpha\beta}^i, \tag{13}$$

using the second equation in (12). For hypersurfaces, there will be only one normal and in such a case, the computation of  $\delta H$  has been given in Ref. [9]. The evaluation of  $\delta H_{\alpha\beta}^i$  for two-dimensional surface in  $d$ -dimensional Riemannian manifold is described in Ref. [13]. The computation of  $\delta H_{\alpha\beta}^i$  for  $m$ -dimensional surface in  $d$ -dimensional Riemannian manifold is involved and we give here the relevant steps for the sake of completeness.

From (7), we have

$$H_{\alpha\beta}^i = \{\partial_\alpha\partial_\beta X^\mu + \tilde{\Gamma}_{\rho\sigma}^\mu\partial_\alpha X^\rho\partial_\beta X^\sigma\}N^{i\nu}h_{\mu\nu}. \tag{14}$$

Using (6), we have

$$\begin{aligned} \delta N^{j\mu}N^{i\nu}h_{\mu\nu} + N^{j\mu}\delta N^{i\nu}h_{\mu\nu} &= -N^{j\mu}N^{i\nu}\delta h_{\mu\nu}, \\ \partial_\gamma(\delta X^\mu)N^{i\nu}h_{\mu\nu} + \partial_\gamma X^\mu\delta N^{i\nu}h_{\mu\nu} &= -\partial_\gamma X^\mu N^{i\nu}\delta h_{\mu\nu}, \end{aligned}$$

and then,

$$\begin{aligned} g^{\alpha\beta}H^i\delta H_{\alpha\beta}^i &= g^{\alpha\beta}H^i\tilde{R}_{\rho\sigma\nu\lambda}\partial_\alpha X^\nu\partial_\beta X^\rho\delta X^\sigma N^{i\lambda} + g^{\alpha\beta}H^i(\nabla_\alpha\nabla_\beta\delta X^\mu)N_\mu^i \\ &\quad + \frac{1}{2}mH^iH^jN^{j\mu}N^{i\nu}(\partial_\lambda h_{\mu\nu})\delta X^\lambda - mH^iH^j\tilde{\Gamma}_{\rho\lambda}^\mu N^{j\sigma}\delta X^\lambda N_\mu^i. \end{aligned} \tag{15}$$

The last two terms cancel each other after expanding  $\tilde{\Gamma}_{\rho\lambda}^\mu$  and using  $i \leftrightarrow j$  symmetry. Now using (8) and (11), we find

$$\begin{aligned} g^{\alpha\beta}H^i\delta H_{\alpha\beta}^i &= g^{\alpha\beta}H^i\tilde{R}_{\rho\sigma\nu\lambda}\partial_\alpha X^\nu\partial_\beta X^\rho\phi^k N^{k\sigma}N^{i\lambda} \\ &\quad + H^k(\nabla_\alpha\nabla^\alpha\phi^k) - H^iH_{\alpha\beta}^iH^{k\alpha\beta}\phi^k. \end{aligned} \tag{16}$$

We consider the extremum of the following functional:

$$W = \int \sqrt{g}(H^iH^i)^{m/2} d^m\xi, \tag{17}$$

which reduces to Willmore functional for  $m = 2$  and to that of Chen [8] for  $m$ -dimensional hypersurface as well with Willmore and Jhaveri [9] for  $m$ -dimensional hypersurface in  $(m + 1)$ -dimensional Riemannian manifold. The normal variations of (17) give the equations of motion. Taking the normal variations of (17) and using (12), (13) and (16), we obtain

$$\begin{aligned} \delta W &= \int \sqrt{g}(H^jH^j)^{(m/2)-1}H^k(\nabla_\alpha\nabla^\alpha\phi^k) d^m\xi - m \int \sqrt{g}\phi^k H^k(H^jH^j)^{m/2} d^m\xi \\ &\quad + \int \sqrt{g}(H^jH^j)^{(m/2)-1}\phi^k H^iH_{\alpha\beta}^iH^{k\alpha\beta} d^m\xi \\ &\quad + \int \sqrt{g}(H^jH^j)^{(m/2)-1}H^i g^{\alpha\beta}\tilde{R}_{\rho\sigma\nu\lambda}\partial_\alpha X^\nu\partial_\beta X^\rho\phi^k N^{k\sigma}N^{i\lambda} d^m\xi. \end{aligned} \tag{18}$$

Equating this to zero and using

$$\int \sqrt{g}(H^j H^j)^{(m/2)-1} H^k (\nabla_\alpha \nabla^\alpha \phi^k) d^m \xi = \int \sqrt{g} \phi^k \nabla_\alpha \nabla^\alpha ((H^j H^j)^{(m/2)-1} H^k) d^m \xi, \tag{19}$$

we obtain the equation of motion for (17) as

$$\nabla_\alpha \nabla^\alpha ((H^j H^j)^{(m/2)-1} H^k) - m H^k (H^j H^j)^{m/2} + (H^j H^j)^{(m/2)-1} H^i H_{\alpha\beta}^i H^{k\alpha\beta} + (H^j H^j)^{(m/2)-1} H^i g^{\alpha\beta} \tilde{R}_{\rho\sigma\nu\lambda} \partial_\alpha X^\nu \partial_\beta X^\rho N^{k\sigma} N^{i\lambda} = 0, \tag{20}$$

since (18) must hold for all allowed  $\phi^k$ .

We now consider various cases.

**Case 1** (Hypersurface in Euclidean space). Let  $\Sigma$  be an  $m$ -dimensional hypersurface in  $d = (m + 1)$ -dimensional *Euclidean* space, i.e.,  $\Sigma' = E^{m+1}$ . As there will be only one normal for a hypersurface,  $H^j H^j = H^2$  and (20) reduces to

$$\nabla_\alpha \nabla^\alpha (H^{m-1}) - m H^{m+1} + H^{m-1} H_{\alpha\beta} H^{\alpha\beta} = 0, \tag{21}$$

and in this case the Gauss equation (9) when contracted with  $g^{\alpha\gamma} g^{\beta\delta}$  gives

$$H^{\alpha\beta} H_{\alpha\beta} = -R + m^2 H^2,$$

where  $R$  is the curvature scalar of the hypersurface  $\Sigma$ . Then (21) becomes

$$\nabla_\alpha \nabla^\alpha (H^{m-1}) + m(m - 1) H^{m+1} - H^{m-1} R = 0, \tag{22}$$

which is the result of Chen [8] and agrees with Eq. (5.59) of Willmore [7].

**Case 2** (Hypersurface in Riemannian space). Let  $\Sigma$  be an  $m$ -dimensional *hypersurface* immersed in  $d = (m + 1)$ -dimensional *Riemannian* manifold  $\Sigma'$ . Then Eq. (20) becomes

$$\nabla_\alpha \nabla^\alpha H^{m-1} - m H^{m+1} + H^{m-1} H_{\alpha\beta} H^{\alpha\beta} + H^{m-1} g^{\alpha\beta} \tilde{R}_{\rho\sigma\nu\lambda} \partial_\alpha X^\nu \partial_\beta X^\rho N^\sigma N^\lambda = 0. \tag{23}$$

In this case, the equation of Gauss (9) is

$$R_{\alpha\beta\gamma\delta} = H_{\beta\delta} H_{\alpha\gamma} - H_{\beta\gamma} H_{\alpha\delta} + \tilde{R}_{\mu\nu\rho\sigma} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \partial_\delta X^\sigma,$$

which when contracted with  $g^{\alpha\gamma} g^{\beta\delta}$  gives

$$R = -H^{\alpha\beta} H_{\alpha\beta} + m^2 H^2 + \tilde{R}_{\mu\nu\rho\sigma} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\gamma X^\rho \partial_\delta X^\sigma g^{\alpha\gamma} g^{\beta\delta}. \tag{24}$$

The completeness relation found in Ref. [13] will now be used and it is

$$h^{\mu\nu} = g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + \sum_i^{d-2} N^{i\mu} N^{i\nu}. \tag{25}$$

For hypersurfaces, (25) is simply

$$h^{\mu\nu} = g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + N^\mu N^\nu, \tag{26}$$

and using this in (24), we find

$$H^{\alpha\beta} H_{\alpha\beta} = -R + m^2 H^2 + \tilde{R} - 2\tilde{R}_{\mu\nu} N^\mu N^\nu. \tag{27}$$

Using (26) in the last term of (23), we have

$$g^{\alpha\beta} \tilde{R}_{\rho\sigma\nu\lambda} \partial_\alpha X^\nu \partial_\beta X^\rho N^\sigma N^\lambda = \tilde{R}_{\rho\sigma\nu\lambda} h^{\rho\nu} N^\sigma N^\lambda - \tilde{R}_{\rho\sigma\nu\lambda} N^\nu N^\sigma N^\rho N^\lambda = \tilde{R}_{\sigma\lambda} N^\sigma N^\lambda,$$

and so Eq. (23) becomes

$$\nabla_\alpha \nabla^\alpha H^{m-1} + m(m-1)H^{m+1} + H^{m-1} \{-R + \tilde{R} - \tilde{R}_{\mu\nu} N^\mu N^\nu\} = 0. \tag{28}$$

We analyse (28) by choosing an orthogonal frame at  $P \in \Sigma$  such that the matrix  $H^{\alpha\beta}$  is diagonal. Then  $H^{\alpha\beta} H_{\alpha\beta} = \sum_{i=1}^m h_i^2$ , and  $m^2 H^2 = (\sum_{i=1}^m h_i)^2$  (see [9]). Then, Eq. (28) can be written as

$$\nabla_\alpha \nabla^\alpha H^{m-1} = -H^{m-1} \left\{ \sum_{i=1}^m h_i^2 - \frac{1}{m} \left( \sum_{i=1}^m h_i \right)^2 + \tilde{R}_{\mu\nu} N^\mu N^\nu \right\}, \tag{29}$$

since (27) gives

$$-R + \tilde{R} + m^2 H^2 - 2\tilde{R}_{\mu\nu} N^\mu N^\nu = \sum_{i=1}^m h_i^2.$$

It is to be noted that  $\sum_{i=1}^m h_i^2 - (1/m)(\sum_{i=1}^m h_i)^2 \geq 0$ . For  $\tilde{R}_{\mu\nu}$  positive-definite, it is seen from (29) that  $\nabla_\alpha \nabla^\alpha H^{m-1}$  has the same sign as  $-H^{m-1}$ .

**Case 3** (Two-dimensional surface in Riemannian space). Let  $\Sigma$  be a two-dimensional surface immersed in  $\Sigma'$ . Then (20) becomes

$$\nabla_\alpha \nabla^\alpha H^k - 2H^k (H^j H^j) + H^i H_{\alpha\beta}^i H^{k\alpha\beta} + H^i g^{\alpha\beta} \tilde{R}_{\rho\sigma\nu\lambda} \partial_\alpha X^\nu \partial_\beta X^\rho N^{k\sigma} N^{i\lambda} = 0, \tag{30}$$

agreeing with Eq. (35) of Ref. [13].

**Case 4** (Immersion in space of constant curvature). Consider  $\Sigma'$  space to be a space of constant curvature, i.e., de-Sitter or anti-de-Sitter-type. In this case [14]

$$\tilde{R}_{\mu\nu\rho\sigma} = K(h_{\mu\rho} h_{\nu\sigma} - h_{\mu\sigma} h_{\nu\rho}). \tag{31}$$

Then,

$$\tilde{R}_{\nu\sigma} = h^{\mu\rho} \tilde{R}_{\mu\nu\rho\sigma} = K(d-1)h_{\nu\sigma}, \quad \tilde{R} = h^{\mu\sigma} \tilde{R}_{\mu\sigma} = Kd(d-1). \tag{32}$$

Then (20) becomes

$$\nabla_\alpha \nabla^\alpha ((H^j H^j)^{(m/2)-1} H^k) + m H^k (H^j H^j)^{(m/2)-1} (K - H^\ell H^\ell) + (H^j H^j)^{(m/2)-1} H^i H_{\alpha\beta}^i H^{k\alpha\beta} = 0. \tag{33}$$

Similarly, Eq. (28) in Case 2 becomes

$$\nabla_\alpha \nabla^\alpha H^{m-1} + m(m-1)H^{m+1} - R H^{m-1} = 0 \tag{34}$$

for  $m$ -dimensional hypersurface immersed in  $\Sigma'$  space of constant curvature.

Eq. (30) of Case 3, i.e., two-dimensional surface immersed in  $\Sigma'$ , a space of constant curvature, becomes

$$\nabla_\alpha \nabla^\alpha H^k - 2H^k (H^j H^j) + H^i H^{i\alpha\beta} H_{\alpha\beta}^k + 2KH^k = 0. \tag{35}$$

In spite of the simplifications, these equations are still difficult to solve explicitly without further choices for the geometry.

### 3. QCD string and Willmore functional

As explained in Section 1, it appears that a candidate action for describing QCD string has to involve the extrinsic geometry of the world sheet, regarded as a two-dimensional Riemannian surface immersed in  $R^4$ . With *negative* stiffness, Kleinert and Chervyakov [5,22] successfully obtained the correct high temperature behaviour as in large- $N$  QCD [6]. Further evidence for the role of the extrinsic geometry in QCD stems from the  $U(N)$ ,  $N \rightarrow \infty$ , lattice gauge theory calculations of Kazakov [15], Kostov [16] and O'Brien and Zuber [17]. These calculations confirm the equivalence of multicolour QCD and string theory in which the resulting surfaces intersect at self-intersections. It is known that the self-intersection number involves extrinsic geometry. We consider the action (4) without the Euler characteristic term as

$$S_{KC} = T \int \sqrt{g} d^2\xi + \alpha_0 \int \sqrt{g} |H|^2 d^2\xi, \tag{36}$$

where  $T$  is the string tension,  $\alpha_0$  a measure of stiffness of the QCD string immersed in  $R^d$  (say) and  $H^2 = H^i H^i$ ,  $i = 1, 2, \dots, (d-2)$ . The extremum of (36) can be easily found using (12), (13) and (16) for normal variations. We find the equation of motion for (36) as

$$\nabla_\alpha \nabla^\alpha H^k - \frac{2T}{\alpha_0} H^k - 2H^k H^j H^j + H^i H^{k\alpha\beta} H_{\alpha\beta}^i = 0 \tag{37}$$

for the stiffness parameter  $\alpha_0 \neq 0$ . This non-linear equation is complicated and it will be worthwhile hence to draw some information from this.

Kholodenko and Nesterenko [18] proposed an approach in this direction by considering (36) for immersion in  $R^3$  and relating to the extremum of the Willmore functional for



immersion in  $S^3$ . We will generalize this approach here by relating (37) to (20) for  $m = 2$ , which is (30). Eq. (30) has the same form as (37) provided we identify,

$$-\frac{2T}{\alpha_0} H^k = H^i \tilde{R}_{\rho\sigma\nu\lambda} \partial_\alpha X^\nu \partial_\beta X^\rho N^{k\sigma} N^{i\lambda} g^{\alpha\beta}. \quad (38)$$

Upon using (25), this becomes

$$-\frac{2T}{\alpha_0} H^k = H^i (\tilde{R}_{\sigma\lambda} N^{k\sigma} N^{i\lambda} - \tilde{R}_{\rho\sigma\nu\lambda} N^{j\nu} N^{j\rho} N^{k\sigma} N^{i\lambda}), \quad (39)$$

a *new relation* among the string tension, stiffness parameter, mean curvature scalar and the geometric properties of  $\Sigma'$ . In order to make (39) manageable, we take  $\Sigma'$  a space of constant curvature as in (31) and (32). Then it can be seen,

$$\tilde{R}_{\sigma\lambda} N^{k\sigma} N^{i\lambda} = K(d-1)\delta_{ik}, \quad \tilde{R}_{\rho\sigma\nu\lambda} N^{j\nu} N^{j\rho} N^{k\sigma} N^{i\lambda} = K(d-3)\delta^{ik}, \quad (40)$$

and so (39) becomes

$$-\frac{2T}{\alpha_0} H^k = K(d-1)H^k - K(d-3)H^k = 2KH^k. \quad (41)$$

Now as we have assumed that  $H^k$ 's are not zero, it follows:

$$\frac{T}{\alpha_0} = -K. \quad (42)$$

It is noted here that the dimensionality of  $\Sigma'$  does not directly appear in relating  $T$ , the string tension, and  $\alpha_0$ , the stiffness parameter, with  $K$ . From this, it follows that the stiffness parameter can be positive for  $K < 0$  (anti-de-Sitter background) or negative for  $K > 0$  (de-Sitter background).

#### 4. Conclusions

The extremum of the Willmore functional for  $m$ -dimensional surface immersed in  $d$ -dimensional Riemannian space is studied under the normal variations of the immersed surface. Various cases of interest are examined. In particular, the equation of motion for a two-dimensional surface immersed in spaces of constant curvature, is compared with the equation of motion of the Polyakov–Kleinert action of the QCD string considered as a Riemann surface immersed in  $R^4$ , to obtain a new relation connecting the string tension and stiffness parameter of the QCD string on the one hand and the constant  $K$  of the Riemann space (31). This relation  $T/\alpha_0 = -K$ , is independent of the dimensionality of  $\Sigma'$ . For positive  $K$ , favoured by positive-definiteness of  $\tilde{R}_{\mu\nu}$  (see Case 2) from (32), it follows that negative stiffness is recommended by geometric considerations. This result agrees with the observation of Kleinert and Chervyakov [5,22] using (physical) QCD string. Thus the QCD string world sheet regarded as a two-dimensional surface immersed in  $R^4$  has been shown to favour negative stiffness by comparing its classical equation with that of a Willmore two-dimensional surface immersed in a space of constant curvature.

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